THE INTEGRAL REPRESENTATION OF FRACTIONALLY EXPONENTIAL FUNCTIONS AND THEIR APPLICATION TO DYNAMIC PROBLEMS OF LINEAR VISCO-ELASTICITY

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Fractionally exponential functions are written in the integral form and distribution functions with an Abelian singularity are obtained for the corresponding relaxation and retardation spectra. A principle is stated, defining the dynamic problems for which weakly singular functions can be used as the kernels of the integral operators. A one-dimensional sound wave traveling in a semiinfinite visco-elastic medium is considered. The generalized exponential functions of fractional order, proposed by Yu. N. Rabotnov [1, 2] as the kernels of Boltzmann-Volterra integral relations, have found wide applications in theory of linear visco-elasticity. This is explained partly by the great mathematical flexibility of the F-operators when applying the Volterra principle to the solution of elastically hereditary problems and partly by the fact that almost all weakly singular kernels possessing an Abelian singularity are connected in some way or other with the F-functions. For example, the resolvent of the elementary weakly singular Abelian kernel is an F-function. The product of an exponential function with an Abelian kernel represents a particular case of the product of two F-functions with different fractional parameters, while the resolvent of such a kernel is the product of an exponential function with an F-function [3, 4]. Since the ε-functions are defined by slowly convergent series, their various asymptotic forms [2, 5-8] are commonly used in practical calculations. The theory of F-functions can be developed further in the context of their integral representations, which enables a more exact physical interpretation to be given to their parameters and on occasion simplifies computational operations.

1. The most general definition of F_γ-functions is given in [1]; while the same approach will be used here, the working will be performed in Laplace space and different notation is introduced.

The following relations between the stress σ and deformation ε are taken as fundamental:

$$\sigma = E_\infty\left[\varepsilon(t) - \nu_\varepsilon \int_0^\infty R(t')\varepsilon(t-t')\,dt'\right],$$

$$\nu_\varepsilon = (E_\infty - E_0)E_\infty^{-1} \tag{1.1}$$

$$\varepsilon = J_\infty\left[\sigma(t) + \nu_\sigma \int_0^\infty K(t')\sigma(t-t')\,dt'\right],$$

$$\nu_\sigma = (J_0 - J_\infty)J_\infty^{-1}.$$

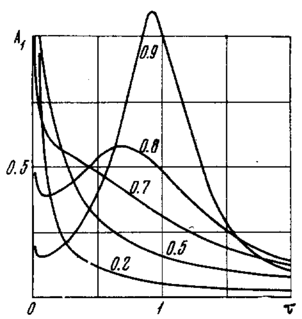

Fig. 1

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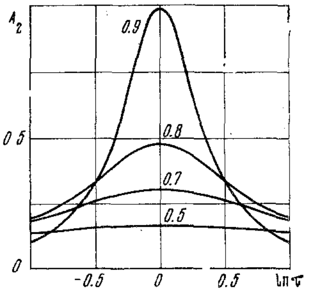

Fig. 2

Here $E_\infty = J_\infty^{-1}$, $E_0 = J_0^{-1}$ are respectively the nonrelaxation and relaxation values of the elastic modulus and pliability, while R(t) and K(t) are the relaxation and after-effect kernels, which are expressible in terms of the distributions $A_1(\tau, \tau_\varepsilon)$ and $B_1(\tau, \tau_\sigma)$ of the relaxation time τ_ε and retardation time τ_σ respectively

$$R(t) = \int_0^\infty \tau^{-1} A_1(\tau,\ \tau_\varepsilon)\, e^{-t/\tau} d\tau, \quad K(t) = \int_0^\infty \tau^{-1} B_1(\tau,\ \tau_\sigma)\, e^{-t/\tau} d\tau. \qquad (1.2)$$

The expression connecting the Laplace transformants $R_*(p)$ and $K_*(p)$ of the relaxation and after-effect kernels is

$$E_\infty - E_0 = E_\infty R_*^{-1}(p) - E_0 K_*^{-1}(p). \qquad (1.3)$$

The simplest hereditary function possessing an integrable singularity is the Abel kernel, which characterizes the unsteady process. It can therefore be used meaningfully as the after-effect kernel for describing unsteady creep [9]. But formally, there are two possibilities

$$K(t) = t^{\gamma-1} / \Gamma(\gamma) \tau_\sigma^\gamma, \quad R(t) = t^{\gamma-1} / \Gamma(\gamma) \tau_\varepsilon^\gamma, \quad E_0 \tau_\sigma^\gamma = E_\infty \tau_\varepsilon^\gamma. \qquad (1.4)$$

Here $\Gamma(\gamma)$ is the gamma function, and γ its parameter of singularity ($0 < \gamma \leq 1$). The following is then obtained from (1.3) for the corresponding resolvents in Laplace space:

$$R_*(p) = \nu_\varepsilon^{-1} [\nu^{-1} (p\tau)^\gamma + 1]^{-1}, \qquad K_*(p) = \nu_\sigma^{-1} [\nu^{-1} (p\tau)^\gamma - 1]^{-1}. \qquad (1.5)$$

Here and below, we take $\tau = \tau_\varepsilon$ if $\nu = \nu_\varepsilon$, and $\tau = \tau_\sigma$ if $\nu = \nu_\sigma$, where τ and ν appear without subscripts.

There are two ways of transforming to the space of originals: first, by formal expansion of the right sides of (1.5) in powers of $\nu(p\tau)^{-\gamma}$, followed by term-by-term passage to the original, and second, by direct application of the Mellin-Fourier inversion formula. In the first case,

$$R(t) = \tau_\varepsilon^{-\gamma} F_\gamma(-\nu,\ t/\tau), \quad K(t) = \tau_\sigma^{-\gamma} F_\gamma(\nu,\ t/\tau),$$
$$F_\gamma(\pm\nu,\ t/\tau) \equiv t^{\gamma-1} \sum_{n=0}^{\infty} \frac{(\pm\nu)^n (t/\tau)^{\gamma n}}{\Gamma[\gamma(n+1)]}. \qquad (1.6)$$

Here F_γ is the Rabotnov fractionally exponential function [1, 2], which is seen from (1.6) to be defined by either an ordinary or an alternating series.

When the Mellin-Fourier formula is applied directly, an integral form is obtained for the F_γ-function

$$R(t) = \tau_\varepsilon^{-\gamma} F_\gamma(-\nu,\ t/\tau) = \frac{\nu_\varepsilon^{-1}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp(pt)\, dp}{1 + \nu^{-1}(p\tau)^\gamma}. \qquad (1.7)$$

Here the case when the minus sign appears in front of the parameter ν of the F_γ-function is taken for clarity, since this is the meaningful case as regards applications.

When $\gamma \neq 1$, the singular points of the integrand of (1.7) are the branch points $p = 0$ and $p = \infty$, and the simple poles at the p for which the denominator $(p\tau)^\gamma + \nu$ vanishes. The latter are

$$p_{1,2} = \tau^{-1} \nu^{1/\gamma} [\cos(\pi/\gamma) \pm i \sin(\pi/\gamma)]. \qquad (1.8)$$

The inversion theorem can only be applied to many-valued functions having branch points on the first sheet of the Riemann surface, i.e., with $-\pi \leq \arg p \leq \pi$, and the residues at the points $p_{1,2}$ are discounted when evaluating the line integral (1.7). On the other hand, when $\gamma = 1$, the one singular point $p = \nu\tau^{-1}$ is a pole of the first order, and the integral is evaluated simply by finding the residue at this point. Taking a contour of integration with a cut along the negative real axis and applying Cauchy's theorem, we get

$$\tau_\varepsilon^{-\gamma} F_\gamma(-\nu_\varepsilon,\ t/\tau_\varepsilon) = \frac{\sin\pi\gamma}{\pi\nu_\varepsilon}\int_0^\infty \frac{\tau^{-2}\exp(-t/\tau)\,d\tau}{(\tau/\tau_\varepsilon)^\gamma \nu_\varepsilon + (\tau/\tau_\varepsilon)^{-\gamma}\nu_\varepsilon^{-1} + 2\cos\pi\gamma} \quad (\gamma \neq 1)$$
$$F_1(-\nu,\ t/\tau) = \exp(-\nu t/\tau) \qquad (\gamma = 1)\,. \tag{1.9}$$

Comparing (1.2) and (1.9), the distribution of the relaxation time can be written in the explicit form

$$A_1(\tau,\ \tau_\varepsilon) = \frac{\sin\pi\gamma}{\pi\tau\nu_\varepsilon}\left[\left(\frac{\tau}{\tau_\varepsilon}\right)^\gamma \nu_\varepsilon + \left(\frac{\tau}{\tau_\varepsilon}\right)^{-\gamma}\nu_\varepsilon^{-1} + 2\cos\pi\gamma\right]^{-1} \tag{1.10}$$

When τ is small, the asymptotic expression

$$A_1(\tau,\ \tau_\varepsilon) = \frac{\sin\pi\gamma}{\pi\tau_\varepsilon^\gamma}\tau^{\gamma-1} \tag{1.11}$$

is obtained, which is none other than the distribution for the Abel kernel.

For, on substituting (1.11) in (1.2) and writing the gamma function in Euler's integral form, (1.4) is obtained. In short, corresponding to Rabotnov's relaxation kernel (1.6), with an Abelian singularity at t = 0, we have the distribution (1.10), likewise with an Abelian singularity, though now with respect to the relaxation time at $\tau = 0$. It may easily be seen that the distribution function $A_1(\tau,\tau_\varepsilon)$ tends to zero as $\tau \to \infty$, and has a maximum provided that $\gamma \geq \sin\pi\gamma$, i.e., $\gamma \gtrsim 0.738$. This is seen clearly in Fig. 1, where the figures on the curves refer to the values of the parameter γ, and it is assumed that $\nu_\varepsilon = 1$, $\tau_\varepsilon = 1$. It should be mentioned that, when $\nu_\varepsilon = 1$, the properties of the F_γ-functions are such that the relaxation kernel and its resolvent (the after-effect kernel) can be written in a symmetric form, namely,

$$R(t) = \tau_\varepsilon^{-\gamma} F_\gamma(-1,\ t/\tau_\varepsilon), \qquad K(t) = \tau_\sigma^{-\gamma} F_\gamma(-1,\ t/\tau_\sigma)\,. \tag{1.12}$$

An important property of weakly singular distributions of the relaxation time is that the corresponding distributions of the relaxation time logarithms have no singularity. For, on substituting $s = \ln \nu_\varepsilon^{1/\gamma} \cdot \tau\tau_\varepsilon^{-1}$ in (1.2) and (1.10), (1.11), the respective expressions

$$A_2(s) = [2\pi\,(\mathrm{ch}\,\gamma s + \cos\pi\gamma)]^{-1}\sin\pi\gamma, \qquad \nu_\varepsilon A_2(s) = \pi^{-1}\sin(\pi\gamma)\exp(\gamma s) \tag{1.13}$$

are obtained.

The first of (1.13) was obtained in [10] when investigating the dispersion of a dielectric constant. It is clear from Fig. 2, where the numbers on the curves again refer to the value of γ, that the graph of $A_2(s)$ is a symmetrical hump, with a maximum at s = 0 equal to

$$A_2(0) = {}^1/_2\,\pi^{-1}\,\mathrm{tg}\,\psi, \qquad \psi = {}^1/_2\,\pi\gamma\,.$$

2. Consider the behavior of the elastically hereditary relationships (1.1) in the case of harmonic deformation (stress). On writing (1.1) in Fourier space ($p \to i\omega$), we then obtain

$$w = \frac{1}{z^*}, \qquad w = x_\varepsilon + iy_\varepsilon, \qquad z^* = x_\sigma - iy_\sigma\,. \tag{2.1}$$

Here $w = E/E_\infty$ and $z^* = J/J_\infty$ are the complex values of the modulus and pliability respectively, their real and imaginary parts being given by

$$x_\varepsilon = 1 - \nu_\varepsilon\int_0^\infty R(t)\cos\omega t\,dt, \qquad y_\varepsilon = \nu_\varepsilon\int_0^\infty R(t)\sin\omega t\,dt \tag{2.2}$$

$$x_\sigma = 1 + \nu_\sigma\int_0^\infty K(t)\cos\omega t\,dt, \qquad y_\sigma = \nu_\sigma\int_0^\infty K(t)\sin\omega t\,dt\,. \tag{2.3}$$

In other words, for the elastically hereditary media (1.1), transformation from the pliability to the modulus is possible via the function of a complex variable (2.1). Since the introduction of the $\mathcal{E}_\gamma$-functions

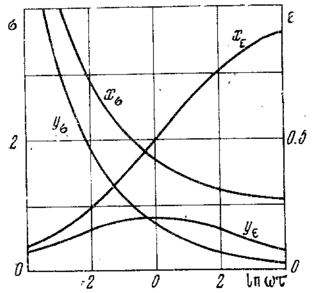

Fig. 3

involves finding the resolvent kernels, it seems worth discovering the quantities which are resolvent invariants, i.e., remain unchanged in the Fourier space on mapping from the plane of the complex pliability to the plane of the complex modulus. From (2.1), there are in general two independent resolvent invariants. First we have the ratio of the imaginary to real parts of the modulus and pliability, or the tangent of the so-called angle of mechanical loss

$$\operatorname{tg} \varphi = y_j / x_j, \quad j = \varepsilon, \sigma, \qquad \varphi \in [0, {}^1\!/_2\pi]. \tag{2.4}$$

Second, it is easily seen from (2.1) that the expression

$$(x_\sigma^2 + y_\sigma^2)(x_\varepsilon^2 + y_\varepsilon^2) = 1 \tag{2.5}$$

is invariant.

In particular cases there may be more than two invariants.

In the light of these general remarks, consider how the complex planes of the modulus and pliability behave when obtaining the F_γ-functions.

For Abel's after-effect kernel (1.4) and the corresponding Rabotnov relaxation kernel (1.6), we have

$$x_\sigma = 1 + \nu(\omega\tau)^{-\gamma}\cos\psi, \qquad y_\sigma = \nu(\omega\tau)^{-\gamma}\sin\psi \tag{2.6}$$

$$x_\varepsilon = \frac{\nu^{-1}(\omega\tau)^{-\gamma} + \cos\psi}{\nu^{-1}(\omega\tau)^{\gamma} + \nu(\omega\tau)^{-\gamma} + 2\cos\psi}, \qquad y_\varepsilon = \frac{\sin\psi}{\nu^{-1}(\omega\tau)^{\gamma} + \nu(\omega\tau)^{-\gamma} + 2\cos\psi}. \tag{2.7}$$

The dispersions of (2.6) and (2.7) are illustrated in Fig. 3, where the parameter value $\gamma = {}^1\!/_2$ is taken. The maximum of y_ε is reached when $\nu^{-1}(\omega\tau)^\gamma = 1$ and is equal to $y_\varepsilon^{(m)} = {}^1\!/_2 \operatorname{tg} {}^1\!/_2\psi$.

Equations connecting x_j and y_j are easily obtained from (2.6) and (2.7)

$$y_\sigma = (x_\sigma - 1)\operatorname{tg}\psi, \qquad (x_\varepsilon - {}^1\!/_2)^2 + (y_\varepsilon + {}^1\!/_2 \operatorname{ctg}\psi)^2 = {}^1\!/_4 \csc^2\psi, \tag{2.8}$$

In the (x_σ, y_σ) plane of the complex pliability a straight line is obtained, cutting the axis of abscissae at the point $x_\sigma = 1$ at an angle $2\pi - \psi$, $\psi = {}^1\!/_2\pi\gamma$, measured in the positive direction (counterclockwise).

In the plane $(x_\varepsilon, y_\varepsilon)$ of the complex modulus, the equation of a circle is obtained, with center at

$$x_\varepsilon^c = {}^1\!/_2, \quad y_\varepsilon^c = -{}^1\!/_2 \operatorname{ctg}\psi$$

and passing through the origin, since its radius is

$$r = {}^1\!/_2 \csc\psi. \tag{2.9}$$

When $\gamma = 1$, $\psi = {}^1\!/_2\pi$, the behavior of the complex modulus corresponds to Maxwell's rheological model.

This fact is illustrated in Fig. 4a, where the abscissa represents the real x_j, and the ordinate the imaginary y_j. The numbers on the curves refer to the values of γ. The point $x_j = 1$ corresponds to $\omega = \infty$.

It can be verified by direct substitution of (2.6) and (2.7) in (2.4) and (2.5) that the relevant quantities are invariant, the tangent of the angle of mechanical loss being equal to

$$\operatorname{tg}\varphi = [\cos\psi + \nu^{-1}(\omega\tau)^\gamma]^{-1}\sin\psi. \tag{2.10}$$

The expression (2.10) was used in [11] for describing the background of internal relaxation friction and its physical characteristics and, in particular, for finding the true value of the activation energy.

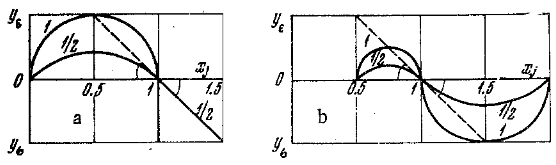

Fig. 4

It should be mentioned that there is a third invariant in the case of kernels with an Abelian singularity, namely the tangent $\operatorname{tg}\psi$ of the angle at which the tangent to the curve in the complex plane cuts the real axis at $\omega = \infty$, i.e., the absolute value of the angle ψ remains unchanged at the point corresponding to $\omega = \infty$.

All in all, it can be said that the Abelian singularity of hereditary kernels with respect to time at $t = 0$, and of the corresponding distribution functions with respect to the retardation (relaxation) time at $\tau = 0$, defines in the complex planes of the pliability and modulus the angle $\psi = {}^1\!/_2\pi\gamma$ between the tangent to the imaginary curve and the real axis at the frequency $\omega = \infty$. If $\gamma = 1$ there is no singularity and we get, e.g., Maxwell's model with $\psi = {}^1\!/_2\pi$. Since the dimensionless parameter $\omega\tau$ appears when solving most problems in theory of linear visco-elasticity, an Abelian singularity can only be discovered via the parameter γ, which usually appears as the power in $(\omega\tau)^\gamma$. In cases where τ and ω feature independently of one another, the Abelian singularity reveals itself directly at $\tau = 0$ or $\omega = \infty$. This fact is exemplified below by the case of a sound wave traveling in an elastically hereditary medium.

If the symmetric expressions (1.2) for the relaxation and after-effect kernels in terms of the F_γ-functions are used, the complex values of the pliability and modulus may be written as

$$x_j = 1 \pm \nu_j \frac{(\omega\tau_j)^{-\gamma} + \cos\psi}{(\omega\tau_j)^{\gamma} + (\omega\tau_j)^{-\gamma} + 2\cos\psi}, \quad y_j = \frac{\nu_j \sin\psi}{(\omega\tau_j)^{\gamma} + (\omega\tau_j)^{-\gamma} + 2\cos\psi}. \tag{2.11}$$

The dispersions of the quantities (2.11) are given in [12, 13].

On eliminating the parameter $\omega\tau$ from (2.11), the equation of a circle is obtained in both cases.

$$[x_j - (1 \pm {}^1\!/_2\nu_j)]^2 + (y_j + {}^1\!/_2\,\nu_j \operatorname{ctg}\psi)^2 = {}^1\!/_4\nu_j^2\csc^2\psi. \tag{2.12}$$

The circle (2.12) of radius $r_\sigma = {}^1\!/_2\nu_\sigma \csc\psi$, with center at

$$x_\sigma^c = 1 + {}^1\!/_2\,\nu_\sigma, \quad y_\sigma^c = -{}^1\!/_2\,\nu_\sigma \operatorname{ctg}\psi$$

cuts the real x_σ axis at the points

$$x_\sigma(\infty) = 1, \quad x_\sigma(0) = 1 + \nu_\sigma = J_0 J_\infty^{-1}.$$

The circle (2.12) of radius $r_\varepsilon = {}^1\!/_2\nu_\varepsilon \csc\psi$, with center at

$$x_\varepsilon^c = 1 - {}^1\!/_2\,\nu_\varepsilon, \quad y_\varepsilon^c = -{}^1\!/_2\,\nu_\varepsilon \operatorname{ctg}\psi$$

cuts the real x_ε axis at the points

$$x_\varepsilon(0) = 1 - \nu_\varepsilon = E_0 E_\infty^{-1}, \quad x_\varepsilon(\infty) = 1.$$

This means that $\operatorname{tg}(\pi - \psi) = \operatorname{tg}(2\pi - \psi)$, i.e., $|\psi| = \text{invar}$. This is clear from Fig. 4b, where we take $E_0 E_\infty^{-1} = \gamma = {}^1\!/_2$. It is clearly seen that, as $E_0 E_\infty^{-1} \to 0$, Fig. 4b transforms into Fig. 4a.

Substitution of (2.11) in (2.4) and (2.5) confirms that the first two invariants are correct, while

$$\operatorname{tg}\varphi = [E_\infty(\omega\tau_\varepsilon)^\gamma + E_0(\omega\tau_\varepsilon)^{-\gamma} + (E_\infty + E_0)\cos\psi]^{-1}(E_\infty - E_0)\sin\psi. \tag{2.13}$$

Expression (2.13) was used in [14] for describing the peak of internal friction when the logarithm of the relaxation time has a symmetric spectrum.

Due to the symmetry of the chosen relaxation and after-effect kernels, the modulus of the angle ψ remains unchanged when $\omega = 0$, as well as when $\omega = \infty$. This fact results in extra resolvent invariants making their appearance. Examples include the ratios of the lengths of the circles to the modulus and pliability deficiencies, the ratios of the areas of the circles, and the ratios of the corresponding segments in the complex half-planes to the squares of the modulus and pliability deficiencies.

3. As an example illustrating the application of F_γ-functions to dynamic problems of hereditary elasticity, take the propagation of a one-dimensional plane tranverse sound wave in the positive direction of the x axis. In view of (1.1), the equation of motion can be written in the two equivalent forms

$$\begin{aligned} \ddot{u} &= c_\infty^2 \frac{\partial^2}{\partial x^2}\left[u - \nu_\varepsilon \int_0^\infty R(t')\,u(t - t', x)\,dt'\right] \\ \ddot{u} &= c_\infty^2 \frac{\partial^2 u}{\partial x^2} - \nu_\sigma \int_0^\infty K(t')\,u(t - t', x)\,dt'. \end{aligned} \tag{3.1}$$

Here u = u(x,t) is the displacement vector in the direction perpendicular to the x axis, while the dots denote differentiation with respect to time.

For the stationary solution

$$u(x, t) = X(x)\, e^{i\omega t} \tag{3.2}$$

the following equation is obtained from (3.1) for the function X(x):

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad \lambda^2 = c_\infty^{-2}\omega^2 w^{-1} = c_\infty^{-2}\omega^2 z^* . \tag{3.3}$$

The solution of (3.3) is

$$X(x) = Ae^{i\lambda x} + Be^{-i\lambda x} , \tag{3.4}$$

The constants A and B are found from the boundary conditions. It is easily shown that, for a wave damped at x ≥ 0, we have A = 0, B = X(0). The solution of (3.2) is thus

$$u(x, t) = X(0) \exp [i(\omega t - kx) - \alpha x] \tag{3.5}$$

$$k = c_\infty^{-1}\omega\, |z|^{1/2} \cos {}^1\!/_2\, \varphi = c_\infty^{-1}\omega\, |w|^{-1/2} \cos {}^1\!/_2\, \varphi \tag{3.6}$$

$$\alpha = c_\infty^{-1}\omega\, |z|^{1/2} \sin {}^1\!/_2\, \varphi = c_\infty^{-1}\omega\, |w|^{-1/2} \sin {}^1\!/_2\, \varphi . \tag{3.7}$$

Here, k is the modulus of the wave vector, α the coefficient of absorption, and φ the phase-shift between the stress and deformation, given by (2.4).

Equations (3.6) give the phase velocity c and the logarithmic decrement Δ, defining the wave attenuated in space

$$c = \omega k^{-1} = c_\infty |z|^{-1/2} \sec {}^1\!/_2\, \varphi = c_\infty |w|^{1/2} \sec {}^1\!/_2\, \varphi \tag{3.8}$$

$$\Delta = \ln [u(x, t) / u(x + 2\pi\omega^{-1} c, t)] = 2\pi\, \mathrm{tg}\, {}^1\!/_2\varphi = 2\pi\alpha k^{-1} . \tag{3.9}$$

The wave characteristics (3.6)-(3.9) illustrate the properties of weakly singular hereditary functions in stationary dynamic problems.

The dispersion of the phase velocity (3.8) and of the logarithmic decrement (3.9), like the tangent of the angle of mechanical loss (2.4), is determined by the parameter $\omega\tau$, and in accordance with what was said in Sec. 2, the Abelian singularity only reveals itself via the fractional coefficient γ. The situation is different for the coefficient of absorption α (3.7) and the modulus k of the wave vector (3.6), in which the frequency ω appears separately as a factor as well as in the product $\omega\tau$. Hence, for the weakly singular hereditary functions considered here, α and k increase indefinitely as $\omega \to \infty$; this is always true for k. For, let us write the wave characteristics for the hereditary kernels (1.4) and (1.6), by substituting (2.6), (2.7) and (2.10) in (3.6)-(3.9). As a result,

$$k, \alpha = 2^{-1/2} c_\infty^{-1}\omega \{[1 + 2\nu(\omega\tau)^{-\gamma}\cos\psi + \nu^2(\omega\tau)^{-2\gamma}]^{1/2} \pm [1 + \nu(\omega\tau)^{-\gamma}\cos\psi]\}^{1/2} . \tag{3.10}$$

Here, the plus sign holds for k, and the minus for α. When $\gamma = 1$ (Maxwell model), (3.10) simplifies considerably

$$k, \alpha = 2^{-1/2} c_\infty^{-1}\omega \{[1 + \nu^2(\omega\tau)^{-2}]^{1/2} \pm 1\}^{1/2} . \tag{3.11}$$

The asymptotic behaviors of k and α for high and low frequencies are of interest. When $\gamma \neq 1$, we have

$$\omega \gg 1, \quad k \approx c_\infty^{-1}\omega\, [1 + {}^1\!/_2\, \nu(\omega\tau)^{-\gamma}\cos\psi] \tag{3.12}$$

$$\omega \ll 1, \quad k \approx c_\infty^{-1} (\nu/\tau^\gamma)^{1/2} \omega^{1-1/2\,\gamma} \cos {}^1\!/_2\psi . \tag{3.13}$$

Consequently, k increases indefinitely, roughly linearly, as $\omega \to \infty$, while it tends to zero as $\omega \to 0$.

When $\gamma \neq 1$, the situation is similar for the coefficient of absorption α, the asymptotic expressions for which are

$$\omega \gg 1 \quad \alpha \approx {}^1\!/_2\, c_\infty^{-1} \nu\tau^{-\gamma} \omega^{1-\gamma} \sin\psi \tag{3.14}$$

$$\omega \ll 1 \quad \alpha \approx c_\infty^{-1} (\nu/\tau^\gamma)^{1/2} \omega^{1-1/2\gamma} \sin {}^1/_2 \psi . \tag{3.15}$$

For Maxwell's model [$\gamma = 1$, expression (3.11)], the asymptotic behaviors of k and α are given by

$$(\omega \gg 1) \quad k \approx c_\infty^{-1} \omega [1 + {}^1/_2 (\nu / 2\omega\tau)^2], \quad \alpha \approx {}^1/_2 c_\infty^{-1} \nu\tau^{-1} [1 - {}^1/_2(\nu/2\omega\tau)^2] \tag{3.16}$$

$$(\omega \ll 1) \quad k \approx \alpha \approx c_\infty^{-1} \omega^{1/2} (\nu / 2\tau)^{1/2} \tag{3.17}$$

Thus, with $\gamma = 1$ and as $\omega \to \infty$, the absorption α is frequency-independent to a first approximation and defined by the constant $(2c_\infty \tau)^{-1} \nu$. This essential difference in the behavior of α in the cases $\gamma \neq 1$ and $\gamma = 1$ at high frequencies ω confirms the conclusions of Sec. 2. Experimental study of the dispersion of the coefficient of absorption [15] in a wide frequency range confirms (3.10), i.e., the fact that, as the frequency rises, the coefficient of absorption increases and shows no tendency to become constant.

In the case of kernsls (1.4) and (1.6), the following are obtained for the speed c of wave propagation and the logarithmic decrement Δ:

$$c = \sqrt{2}\, c_\infty \{[1 + 2\nu (\omega\tau)^{-\gamma} \cos \psi + \nu^2 (\omega\tau)^{-2\gamma}]^{1/2} + [1 + \nu (\omega\tau)^{-\gamma} \cos \psi]\}^{-1/2} \tag{3.18}$$

$$\left(\frac{\Delta}{2\pi}\right)^2 = \frac{[1 + 2\nu (\omega\tau)^{-\gamma} \cos \psi + \nu^2 (\omega\tau)^{-2\gamma}]^{1/2} - [1 + \nu (\omega\tau)^{-\gamma} \cos \psi]}{[1 + 2\nu (\omega\tau)^{-\gamma} \cos \psi + \nu^2 (\omega\tau)^{-2\gamma}]^{1/2} + [1 + \nu (\omega\tau)^{-\gamma} \cos \psi]} . \tag{3.19}$$

When $\gamma = 1$, we get

$$c = \sqrt{2}\, c_\infty \{[1 + \nu^2 (\omega\tau)^{-2}]^{1/2} + 1\}^{-1/2}, \quad \Delta = \sqrt{2}\pi\, c c_\infty^{-1} \{[1 + \nu^2 (\omega\tau)^{-2}]^{1/2} - 1\}^{1/2}. \tag{3.20}$$

The asymptotic behaviors of c and Δ at high and low frequencies are given by

$\gamma \neq 1$

$$(\omega \gg 1) \quad c = c_\infty [1 - {}^1/_2 \nu (\omega\tau)^{-\gamma} \cos \psi], \quad \Delta \approx \pi \nu (\omega\tau)^{-\gamma} \sin \psi \tag{3.21}$$

$$(\omega \ll 1) \quad c \approx c_\infty \nu^{-1/2} (\omega\tau)^{\gamma/2} \sec {}^1/_2 \psi, \quad \Delta \approx 2\pi [1 - \nu^{-1} (\omega\tau)^\gamma] \operatorname{tg} {}^1/_2 \psi \tag{3.22}$$

$\gamma = 1$

$$(\omega \gg 1) \quad c \approx c_\infty [1 - {}^1/_8 \nu^2 (\omega\tau)^{-2}], \quad \Delta \approx \pi \nu (\omega\tau)^{-1} \tag{3.23}$$

$$(\omega \ll 1) \quad c \approx c_\infty (2\omega\tau \nu^{-1})^{1/2}, \quad \Delta \approx [1 - \nu^{-1} (\omega\tau)]\, 2\pi . \tag{3.24}$$

Expressions (3.10)-(3.24) confirm the conclusions reached in Sec. 2. A study of the wave characteristics for symmetric ε_γ-functions may be found in [16].

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